Mathematical Methods in Physics HW5

1. For the matrix $M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, show that the standard construction of eigenvalues and eigenvectors agrees with the results of the extremizing $I = (x, Mx)$ subject to $(x, x) = 1$.

$\det[M - \lambda I] = 0 = (-\lambda)[(-\lambda)^2 - 1] - [1(-\lambda)] = (-\lambda)[\lambda^2 - 2] \Rightarrow \lambda_1 = 0, \lambda_2 = \sqrt{2}, \lambda_3 = -\sqrt{2}$

$$Mx_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = \lambda_1 x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} b = 0 \\ a = -c \end{matrix} \Rightarrow x_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$Mx_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = \lambda_2 x_2 = \begin{pmatrix} \sqrt{2}a \\ \sqrt{2}b \\ \sqrt{2}c \end{pmatrix} \Rightarrow \begin{matrix} b = \text{anything} \\ a = c = \frac{1}{\sqrt{2}}b \end{matrix} \Rightarrow x_2 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$$Mx_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = \lambda_3 x_3 = \begin{pmatrix} -\sqrt{2}a \\ -\sqrt{2}b \\ -\sqrt{2}c \end{pmatrix} \Rightarrow \begin{matrix} b = \text{anything} \\ a = c = -\frac{1}{\sqrt{2}}b \end{matrix} \Rightarrow x_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

For $I = (x, Mx)$ subject to $(x, x) - 1 = 0 = J$ we can extremize $K = I - \lambda J$ which is:

$$K = (a\ b\ c)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \lambda(a^2 + b^2 + c^2 - 1) = 2ab + 2bc - \lambda a^2 - \lambda b^2 - \lambda c^2 + \lambda$$

	$\lambda_1 = 0$	$\lambda_2 = \sqrt{2}$	$\lambda_3 = -\sqrt{2}$
$\frac{\partial K}{\partial a} = 2b - 2\lambda a = 0$	$b = 0$	$a = \frac{1}{\sqrt{2}}b$	$a = -\frac{1}{\sqrt{2}}b$
$\frac{\partial K}{\partial b} = 2c - 2\lambda b + 2a = 0$	$a = -c$	$b = anything$	$b = anything$
$\frac{\partial K}{\partial c} = 2b - 2\lambda c = 0$	$b = 0$	$c = \frac{1}{\sqrt{2}}b$	$c = -\frac{1}{\sqrt{2}}b$
Then the eigenvectors are:	$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$

2. Consider the matrix $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find the set of vectors y such that $[M - \lambda I]x = y$, where λ is an eigenvalue of M, has a solution.

The eigenvalues of M are $\lambda_1 = 1$ and $\lambda_2 = -1$ and the corresponding normalized eigenvectors are $x_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $x_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

We need y to be orthogonal to the solutions of $\left[M^{\dagger}-\lambda^{*}I\right]\phi_i=0$. But notice that $M^{\dagger}=M$ hence $\lambda^{*}=\lambda$. So the solutions to this equation are just scalar multiples of the eigenvectors of M. But this means that if we consider $[M-\lambda_1 I]x=y$ then y must be orthogonal to x_1 and hence must be a multiple of x_2, and similarly if we take $[M-\lambda_2 I]x=y$ then y must be orthogonal to x_2 and hence must be a multiple of x_1.

3. Show that a linear operator whose combination with its adjoint, i.e. $A^{\dagger}A$, gives a nonzero scalar factor times the identity, also enjoys the property that eigenvectors associated with distinct eigenvalues will always be orthogonal. And while your at it, go ahead and address the question I asked in my notes but did not mention in class, that is, what about anti-isometric matrices? Are they normal?

If $A^{\dagger}A=nI$ then $\frac{A^{\dagger}}{\sqrt{n}}\frac{A}{\sqrt{n}}=I \Rightarrow \frac{A}{\sqrt{n}}\frac{A^{\dagger}}{\sqrt{n}}=I \;\Rightarrow\; AA^{\dagger}=nI \;\Rightarrow A^{\dagger}A=AA^{\dagger} \;\Rightarrow\; \left[A,A^{\dagger}\right]=0 \;\Rightarrow\; A$ is normal. But normal operators are those that enjoy the orthogonal eigenvectors for distinct eigenvalues property.

As for anti-isometric operators, they don't exist! Consider $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $AA^{\dagger}=-I$. This implies that $AA^{\dagger}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^{*} & c^{*} \\ b^{*} & d^{*} \end{pmatrix}=\begin{pmatrix} aa^{*}+bb^{*} & ac^{*}+bd^{*} \\ ca^{*}+db^{*} & cc^{*}+dd^{*} \end{pmatrix}=\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, however $aa^{*}+bb^{*}\geq 0$ and $cc^{*}+dd^{*}\geq 0$.

4. Consider the matrix $A_0=\begin{pmatrix} 3 & 0 & i\sqrt{8} \\ 0 & 2 & 0 \\ -i\sqrt{8} & 0 & 1 \end{pmatrix}$. Now consider a perturbation given by

$A=\begin{pmatrix} 3 & 0 & i\sqrt{8} \\ 0 & 2 & 0 \\ -i\sqrt{8} & 0 & 1 \end{pmatrix}+\epsilon\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. For the new matrix, determine the eigenvalues up to third order in ϵ and the eigenvectors to first order in ϵ using nondegenerate perturbation theory.

Let's start by finding the eigens of A_o.

$$\det\left[A_0-\lambda^{(0)}I\right]=0=\left(3-\lambda^{(0)}\right)\left(2-\lambda^{(0)}\right)\left(1-\lambda^{(0)}\right)-8\left(2-\lambda^{(0)}\right)=\left(2-\lambda^{(0)}\right)\left[\lambda^{(0)^2}-4\lambda^{(0)}-5\right]$$

$$=\left(2-\lambda^{(0)}\right)\left(\lambda^{(0)}-5\right)\left(\lambda^{(0)}+1\right) \;\Rightarrow\; \lambda_1^{(0)}=2,\lambda_2^{(0)}=5,\lambda_3^{(0)}=-1$$

$$A_0x_1^{(0)}=\lambda_1^{(0)}x_1^{(0)}=2x_1^{(0)} \;\Rightarrow\; \begin{array}{r} 3a+i\sqrt{8}c=2a \\ 2b=2b \\ -i\sqrt{8}a+c=2c \end{array} \Rightarrow \begin{array}{c} a=-i2\sqrt{2}c \\ b=any \\ c=-i2\sqrt{2}a \end{array} \Rightarrow \begin{array}{c} b=any \\ a=c=0 \end{array} \Rightarrow x_1^{(0)}=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$A_0x_2^{(0)}=\lambda_2^{(0)}x_2^{(0)}=5x_2^{(0)} \;\Rightarrow\; \begin{array}{r} 3a+i\sqrt{8}c=5a \\ 2b=5b \\ -i\sqrt{8}a+c=5c \end{array} \Rightarrow \begin{array}{c} 2a=i2\sqrt{2}c \\ b=0 \\ 4c=-i2\sqrt{2}a \end{array} \Rightarrow \begin{array}{c} b=0 \\ a=i\sqrt{2}c \end{array} \Rightarrow x_2^{(0)}=\begin{pmatrix} \frac{i\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$A_0 x_3^{(0)} = \lambda_3^{(0)} x_3^{(0)} = -1 x_3^{(0)} \Rightarrow \begin{array}{r} 3a + i\sqrt{8}c = -a \\ 2b = -b \\ -i\sqrt{8}a + c = -c \end{array} \Rightarrow \begin{array}{c} 4a = -i2\sqrt{2}c \\ b = 0 \\ 2c = i2\sqrt{2}a \end{array} \Rightarrow \begin{array}{c} b = 0 \\ c = i\sqrt{2}a \end{array} \Rightarrow x_3^{(0)} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{i\sqrt{2}}{\sqrt{3}} \end{pmatrix}$$

Now we move on to corrections. Starting with the first order correction to eigenvalues:

$$\lambda_1^{(1)} = \left(x_1^{(0)}, A_1 x_1^{(0)}\right) = (0\ 1\ 0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2$$

$$\lambda_2^{(1)} = \left(x_2^{(0)}, A_1 x_2^{(0)}\right) = \left(\frac{-i\sqrt{2}}{\sqrt{3}}\ 0\ \frac{1}{\sqrt{3}}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{i\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \frac{5}{3}$$

$$\lambda_2^{(1)} = \left(x_2^{(0)}, A_1 x_2^{(0)}\right) = \left(\frac{1}{\sqrt{3}}\ 0\ \frac{-i\sqrt{2}}{\sqrt{3}}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{i\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \frac{7}{3}$$

Now we find the first order correction to eigenvectors:

$$x_1^{(1)} = \frac{\left(x_2^{(0)}, A_1 x_1^{(0)}\right)}{\lambda_1^{(0)} - \lambda_2^{(0)}} x_2^{(0)} + \frac{\left(x_3^{(0)}, A_1 x_1^{(0)}\right)}{\lambda_1^{(0)} - \lambda_3^{(0)}} x_3^{(0)} = \frac{0}{-3}\begin{pmatrix} \frac{i\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} + \frac{0}{3}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{i\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_2^{(1)} = \frac{\left(x_1^{(0)}, A_1 x_2^{(0)}\right)}{\lambda_2^{(0)} - \lambda_1^{(0)}} x_1^{(0)} + \frac{\left(x_3^{(0)}, A_1 x_2^{(0)}\right)}{\lambda_2^{(0)} - \lambda_3^{(0)}} x_3^{(0)} = \frac{0}{3}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{i\frac{\sqrt{2}}{3} - i\sqrt{2}}{6}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \frac{i\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{-i\sqrt{2}}{9\sqrt{3}} \\ 0 \\ \frac{2}{9\sqrt{3}} \end{pmatrix}$$

$$x_3^{(1)} = \frac{\left(x_1^{(0)}, A_1 x_3^{(0)}\right)}{\lambda_3^{(0)} - \lambda_1^{(0)}} x_1^{(0)} + \frac{\left(x_2^{(0)}, A_1 x_3^{(0)}\right)}{\lambda_3^{(0)} - \lambda_2^{(0)}} x_2^{(0)} = \frac{0}{-3}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \frac{-i\frac{\sqrt{2}}{3} + i\sqrt{2}}{-6}\begin{pmatrix} \frac{i\sqrt{2}}{\sqrt{3}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} \frac{2}{9\sqrt{3}} \\ 0 \\ \frac{-i\sqrt{2}}{9\sqrt{3}} \end{pmatrix}$$

With these in hand we can now compute the second and third order corrections to the eigenvalues:

$$\lambda_1^{(2)} = \left(x_1^{(0)}, A_1 x_1^{(1)}\right) = (0\ 1\ 0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\lambda_2^{(2)} = \left(x_2^{(0)}, A_1 x_2^{(1)}\right) = \left(\frac{-i\sqrt{2}}{\sqrt{3}}\ 0\ \frac{1}{\sqrt{3}}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{-i\sqrt{2}}{9\sqrt{3}} \\ 0 \\ \frac{2}{9\sqrt{3}} \end{pmatrix} = \frac{4}{27}$$

$$\lambda_3^{(2)} = \left(x_3^{(0)}, A_1 x_3^{(1)}\right) = \left(\frac{1}{\sqrt{3}}\ 0\ \frac{-i\sqrt{2}}{\sqrt{3}}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{2}{9\sqrt{3}} \\ 0 \\ \frac{-i\sqrt{2}}{9\sqrt{3}} \end{pmatrix} = -\frac{4}{27}$$

and

$$\lambda_1^{(3)} = \left(x_1^{(1)}, [A_1 - \lambda_1^{(1)}]x_1^{(1)}\right) = (0\ 0\ 0)\begin{pmatrix} 1-2 & 0 & 0 \\ 0 & 2-2 & 0 \\ 0 & 0 & 3-2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$\lambda_2^{(3)} = \left(x_2^{(1)}, [A_1 - \lambda_2^{(1)}]x_2^{(1)}\right) = \left(\frac{i\sqrt{2}}{9\sqrt{3}}\ 0\ \frac{2}{9\sqrt{3}}\right)\begin{pmatrix} 1-\frac{5}{3} & 0 & 0 \\ 0 & 2-\frac{5}{3} & 0 \\ 0 & 0 & 3-\frac{5}{3} \end{pmatrix}\begin{pmatrix} \frac{-i\sqrt{2}}{9\sqrt{3}} \\ 0 \\ \frac{2}{9\sqrt{3}} \end{pmatrix} = \frac{4}{243}$$

$$\lambda_3^{(3)} = \left(x_3^{(1)}, [A_1 - \lambda_3^{(1)}]x_3^{(1)}\right) = \left(\frac{2}{9\sqrt{3}}\ 0\ \frac{i\sqrt{2}}{9\sqrt{3}}\right)\begin{pmatrix} 1-\frac{7}{3} & 0 & 0 \\ 0 & 2-\frac{7}{3} & 0 \\ 0 & 0 & 3-\frac{7}{3} \end{pmatrix}\begin{pmatrix} \frac{2}{9\sqrt{3}} \\ 0 \\ \frac{-i\sqrt{2}}{9\sqrt{3}} \end{pmatrix} = -\frac{4}{243}$$

So in the end we have:

$$\lambda_1 = \lambda_1^{(0)} + \lambda_1^{(1)}\epsilon + \lambda_1^{(2)}\epsilon^2 + \lambda_1^{(3)}\epsilon^3 + .. = 2 + 2\epsilon + 0\epsilon^2 + 0\epsilon^3 + .. = 2 + 2\epsilon$$

$$\lambda_2 = \lambda_2^{(0)} + \lambda_2^{(1)}\epsilon + \lambda_2^{(2)}\epsilon^2 + \lambda_2^{(3)}\epsilon^3 + .. = 5 + \frac{5}{3}\epsilon + \frac{4}{27}\epsilon^2 + \frac{4}{243}\epsilon^3 + ..$$

$$\lambda_3 = \lambda_3^{(0)} + \lambda_3^{(1)}\epsilon + \lambda_3^{(2)}\epsilon^2 + \lambda_3^{(3)}\epsilon^3 + .. = -1 + \frac{7}{3}\epsilon - \frac{4}{27}\epsilon^2 - \frac{4}{243}\epsilon^3 + ..$$

5. Check that your results from problem (3) for the eigenvalues agree with what you would obtain by directly determining the eigenvalues as an expansion in ϵ.

$$\det[A - \lambda I] = 0 = det\begin{pmatrix} 3+\epsilon-\lambda & 0 & i\sqrt{8} \\ 0 & 2+2\epsilon-\lambda & 0 \\ -i\sqrt{8} & 0 & 1+3\epsilon-\lambda \end{pmatrix} = 0$$

$$0 = (3+\epsilon-\lambda)(2+2\epsilon-\lambda)(1+3\epsilon-\lambda) - 8(2+2\epsilon-\lambda) = (2+2\epsilon-\lambda)[(3+\epsilon-\lambda)(1+3\epsilon-\lambda) - 8]$$

Which means that one of the eigenvalues is exactly $\lambda = 2 + 2\epsilon$, as obtained for λ_1 using perturbation theory.

For the remaining two, we can factorize the term in square brackets. First cleaning it up:

$$(3+\epsilon-\lambda)(1+3\epsilon-\lambda) - 8 = \lambda^2 + (-4-4\epsilon)\lambda + (3\epsilon^2 + 10\epsilon - 5)$$

Then using the quadratic equation:

$$\lambda_\pm(\epsilon) = \frac{4+4\epsilon \pm \sqrt{(-4-4\epsilon)^2 - 12\epsilon^2 - 40\epsilon + 20}}{2} = 2 + 2\epsilon \pm \sqrt{\epsilon^2 - 2\epsilon + 9}$$

We can now Taylor expand the root to get a power series in ϵ :

$$\lambda_+^{(0)} = \lambda_+(0) = 2 + 3 = 5$$

$$\lambda_+^{(1)} = \frac{d}{d\epsilon}\lambda_+(\epsilon)|_{\epsilon=0} = \left[2 + (\epsilon-1)(\epsilon^2 - 2\epsilon + 9)^{-\frac{1}{2}}\right]_{\epsilon=0} = \frac{5}{3}$$

$$\lambda_+^{(2)} = \frac{1}{2}\frac{d^2}{d\epsilon^2}\lambda_+(\epsilon)|_{\epsilon=0} = \frac{1}{2}\left[(\epsilon^2 - 2\epsilon + 9)^{-\frac{1}{2}} - \frac{1}{2}(\epsilon - 1)(2\epsilon - 2)(\epsilon^2 - 2\epsilon + 9)^{-\frac{3}{2}}\right]_{\epsilon=0} = \frac{4}{27}$$

$$\lambda_-^{(0)} = \lambda_-(0) = 2 - 3 = -1$$

$$\lambda_-^{(1)} = \frac{d}{d\epsilon}\lambda_-(\epsilon)|_{\epsilon=0} = \left[2 - (\epsilon - 1)(\epsilon^2 - 2\epsilon + 9)^{-\frac{1}{2}}\right]_{\epsilon=0} = \frac{7}{3}$$

$$\lambda_-^{(2)} = \frac{1}{2}\frac{d^2}{d\epsilon^2}\lambda_-(\epsilon)|_{\epsilon=0} = \frac{1}{2}\left[-(\epsilon^2 - 2\epsilon + 9)^{-\frac{1}{2}} + \frac{1}{2}(\epsilon - 1)(2\epsilon - 2)(\epsilon^2 - 2\epsilon + 9)^{-\frac{3}{2}}\right]_{\epsilon=0} = -\frac{4}{27}$$

So clearly:

$\lambda_+ = \lambda_2$ and $\lambda_- = \lambda_3$.